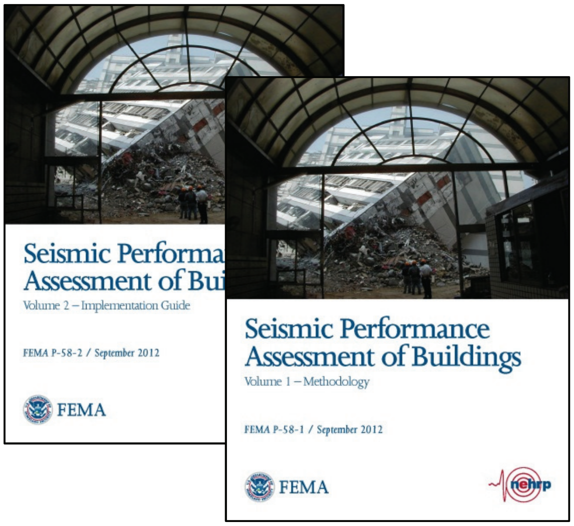

Background Document

FEMA P-58/BD-3.7.10

Generating Statistically Consistent Demand Vectors for Loss Computations using Spectral Value Decomposition

Prepared by

Dhiman Basu and Andrew Whittaker
Department of Civil, Structural, and Environmental Engineering
212 Ketter Hall
University at Buffalo
Buffalo, New York 14260

Submitted to

APPLIED TECHNOLOGY COUNCIL
201 Redwood Shores Parkway, Suite 240
Redwood City, California 94065
www.ATCouncil.org

Prepared for

FEDERAL EMERGENCY MANAGEMENT AGENCY
U.S. Department of Homeland Security
500 C Street, SW
Washington, D.C. 20472

May 27, 2011

FEMA

nehrp

Background Documentation

FEMA P-58 Background Documents are a series of reports documenting the technical background and source information for key aspects of the FEMA P-58 methodology and its implementation. These reports were developed over the course of the 10-year ATC-58/ATC-58-1 Projects funded under FEMA Contracts EMW-2001-RP-0056 and HSFEHQ-06-D-1105.

Background Documents were developed by consultants, serving at various levels within the project hierarchy, reporting the results of: (1) decisions on technical development protocols; (2) focused studies on the development of key aspects of the methodology; (3) documentation of recommended procedures; and (4) collection of available data for the development of structural and nonstructural fragilities. They were initially intended to serve as a record of the technical state-of-knowledge at the time they were produced, and as resources for the development of the eventual project reports. As such, they represent a snapshot in time, and may, or may not, match the technical content, recommended procedures, or data incorporated into the final methodology and its implementation.

This Background Document is intended for the purpose of providing supplemental knowledge to users of the FEMA P-58 methodology. Information contained herein has not been independently verified for accuracy as a stand-alone document, and may have been superseded in its final implementation within the methodology. Users of information in this document assume all liability arising from such use.

Cover illustration – Primary resource documents for the FEMA P-58 *Seismic Performance Assessment of Buildings, Methodology and Implementation* series of products: FEMA P-58-1, *Volume 1 – Methodology*, and FEMA P-58-2, *Volume 2 – Implementation Guide*.

Rev 7
5/27/11

Technical Note

Generating statistically consistent demand vectors for loss computations using spectral value decomposition

Dhiman Basu[1] and Andrew Whittaker[2]

1. Introduction

One product of the ATC-58 project on Performance Based Seismic Design of Buildings (www.atcouncil.org) is a Performance Assessment Calculation Tool (PACT) that performs most of the repetitive calculations required to estimate losses for intensity-, scenario- and time-based assessments. PACT draws upon a number of databases developed as part of the ATC-58 project to compute distributions of loss.

The loss-computation algorithm in PACT generates a large number of realizations (or vectors) of demand per intensity level to develop a loss curve. Yang and his co-workers developed the technical basis for the algorithm implemented in PACT at the time of this writing (e.g., Yang, 2006; Yang et al., 2009). The Yang et al. algorithm was subsequently updated by Huang et al. in the 50% draft Guideline to address modeling uncertainty β_m, record-to-record variability β_a and ground motion variability β_{gm}.

For typical performance assessment, a limited number, *m*, of response analyses are performed at each intensity level. For each analysis, the peak absolute value of each demand parameter (e.g., third story drift, roof acceleration) is assembled into a row vector with *n* entries, where *n* is the number of demand parameters. The *m* row vectors are catenated to form an $m \times n$ matrix (rows × columns; simulations × demand parameters). Each column presents *m* values of one demand parameter. The matrix of demand parameters is denoted $[X]$ in the discussion that follows.

Section 2 describes the algorithm implemented in PACT at the time of publication of the 50% draft Guideline for Performance Based Seismic Design of Buildings and subsequent revisions proposed to the algorithm by Zareian (2010). An alternate algorithm, which is based on spectral value decomposition (SVD), and its coding in Matlab (Mathworks, 2010) is presented in Section 3. Section 4 provides an evaluation of results generated using both PACT and the alternate algorithm. A list of references follows Section 4. Appendix A provides the mathematical basis for the alternate algorithm of Section 3.

[1] PhD candidate, Department of Civil, Structural and Environmental Engineering State University of New York, Buffalo
[2] Professor and Chair, Department of Civil, Structural and Environmental Engineering, State University of New York, Buffalo

2. PACT algorithm for generating statistically consistent demand vectors

The entries in the matrix $[X]$ are assumed to be jointly lognormal. The natural logarithm of each entry in $[X]$ is computed to form an $m \times n$ matrix $[Y]$. The entries in $[Y]$ are assumed to be jointly normal and can be characterized by $1 \times n$ vector of means, $\{m_Y\}$, a $n \times n$ correlation coefficient matrix, $[R_{YY}]$ and a diagonal matrix of standard deviations, $[D_Y]$.

We seek to generate a larger matrix of demand vectors $[Z]$ with the same statistical distribution as $[Y]$. This process uses a set of vectors, one for each demand parameter, of uncorrelated standard normal random variables, $[U]$, with a zero mean and a unit standard deviation, and a linear transformation and a linear translation:

$$[Z]^T = [A][U]^T + [B] = [D_Y][L_Y][U]^T + [M_Y] \tag{1}$$

Matrices $[A]$ and $[B]$ are derived in Yang et al. (2006, 2009). The matrix $[L_Y]$ is the transposed Cholesky decomposition[3] of $[R_{YY}]$, and all other terms are defined above. Information on the Cholesky decomposition of a matrix can be found in textbooks on linear algebra (e.g., Wilkinson, 1965).

Realizations of demand are generated by 1) computing $[M_Y]$, $[D_Y]$ and $[L_Y]$ by sampling $[Y]$; 2) updating $[D_Y]$ for modeling uncertainty [not addressed in this Note], 3) populating $[U]$ by random sampling each demand parameter on a distribution with a mean of 0 and a standard deviation of 1.0; 4) computing $[Z]$ per (1); and 5) taking the exponential of every entry in $[Z]$ to recover the demand parameters. The process is computationally efficient because $[M_Y]$, $[D_Y]$ and $[L_Y]$ are computed once only for each intensity of shaking. The process is illustrated in Figure 1 (from Yang et al. 2006) below.

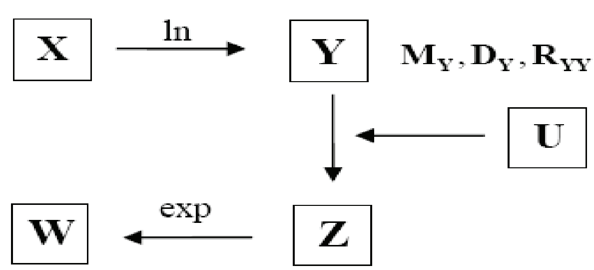

Figure 1: Generation of vectors of correlated demand parameters (Yang 2006)

[3] If matrix $[K]$ is symmetric and positive-definite it can be decomposed into a lower triangular matrix, $[L]$, the Cholesky triangle, and the transpose of the lower triangular matrix, such that $[K] = [L][L]^{\mathrm{T}}$. To solve $[K]\{u\} = [R]$, one solves first $[L]\{y\} = [R]$ for $\{y\}$ and then $[L]^{\mathrm{T}}\{u\} = \{y\}$ for $\{u\}$.

The above algorithm was included in the 50% draft Guidelines. The algorithm was developed by Yang in the Matlab environment. The *chol* subroutine in Matlab was used to perform the decomposition.

Early studies using this algorithm were reported in Appendix G of the 50% draft Guidelines (www.atcouncil.org) using an example that involved 7 demand parameters and 11 ground motions, namely, n = 7 and m = 11. Results reported in Appendix G of that document showed that the algorithm yielded robust results if the simulation space was large, with best results (measured by accurate recovery of statistical properties) being achieved for 1000s of simulations.

Subsequent studies of PACT by others, including Zareian (2010), identified an instability with the *chol* algorithm if the number of demand parameters exceeded the number of ground motions, namely, $n > m$. The solution Zareian proposed was to replace the Matlab *chol* subroutine with the *cholcov* subroutine. The *cholcov* subroutine is stable for both $m \geq n$ and $n > m$. The technical bases for the *chol* and *cholcov* routines are presented in the Matlab documentation and are not repeated here but we note that *chol* is stable only if the associated matrix is not rank deficient whereas *cholcov* is stable for rank deficient matrices.

We provide below the mathematical basis for the statement that if $n > m$, the correlation matrix is rank deficient. We first present an identity of the form (Wilkinson, 1965):

$$\begin{bmatrix} I & o \\ -B & \mu I \end{bmatrix} \begin{bmatrix} \mu I & A \\ B & \mu I \end{bmatrix} = \begin{bmatrix} \mu I & A \\ o & \mu^2 I - BA \end{bmatrix} \tag{2}$$

$$\begin{bmatrix} \mu I & -A \\ o & I \end{bmatrix} \begin{bmatrix} \mu I & A \\ B & \mu I \end{bmatrix} = \begin{bmatrix} \mu^2 I - AB & o \\ B & \mu I \end{bmatrix} \tag{3}$$

If we define the second matrix on left hand side of Eq (2) [and Eq (3)] as Q, from Eq (2):

$$\mu^n \det(Q) = \mu^n \det\left(\mu^2 I - BA\right) \tag{4}$$

Similarly from Eq (3):

$$\mu^n \det(Q) = \mu^n \det\left(\mu^2 I - AB\right) \tag{5}$$

Therefore, the matrix products $[A][B]$ and $[B][A]$ will have the same eigenvalues. Moreover, if $[A]$ is of size $n \times m$ and $[B]$ is of size $m \times n$ then $[A][B]$ and $[B][A]$ will have the same eigenvalues except the product that is of the higher order will have an additional $|m-n|$ zero eigenvalues.

Now, we consider a specific case, namely, $[A] = [B]^T$. The matrix product $[B]^T[B]$ can be considered as the correlation matrix with some scale factor and will have $(n-m)$ zero eigenvalues if $n > m$. Moreover, when $m = n$ the determinant of the correlation matrix will be zero and hence one of the eigenvalues will be zero. Accordingly, the correlation matrix is also rank deficient if $m = n$. Combining this property with the mathematical proof presented above, the correlation matrix will be rank deficient when $n \geq m$.

3. An alternative algorithm for generating statistically consistent demand vectors

An alternative approach to generate (or simulate) demand vectors is outlined below. The approach is based on the spectral value decomposition of the correlation matrix and is termed the SVD algorithm for the purpose of this presentation. The variables and matrices introduced in Sections 1 and 2 are used here.

The matrix $\left[X\right]$ is transformed into the $\left[Y\right]$ space as shown in Figure 1. It is transformed again into another space, $\left[G\right]$, such that all of the demand parameter vectors have a mean of zero. Matrix $\left[G\right]$ is then expanded into a larger dimensional space $\left[\overline{G}\right]$ such that the mean, variance and correlation matrix are preserved. This is performed by generating the required set of independent random variables $\left[\overline{U}\right]$ of the same distribution with zero mean and unit variance followed by a linear transformation.

The procedure differs from PACT in two important ways. First, the factorization of the correlation matrix is based on spectral value decomposition, which is more robust than the Cholesky-type decomposition. Second, a procedure to accelerate the convergence of the means, variances and correlation coefficients to the target values is provided to enable the use of a smaller number of simulations. (Whether the target values are accurate is another matter, which is beyond the scope of this study.)

An alternative algorithm to *cholcov*, denoted hereafter as the SVD algorithm, is presented below. The Matlab code is provided on the following pages. The mathematical proof is presented in Appendix A. We note that, the uniqueness of the solution is not guaranteed because the computation involves the generation of random numbers and, even though spectral values are unique, by definition, the associated left and right basis vectors are not unique. However, uniqueness is not an issue for the loss computation.

SVD algorithm

1. Read $\left[X\right]$, select OPT= 0 for no acceleration (see above) and 1 for acceleration
2. Compute $\left[Y\right]=\ln\left[X\right]$
3. Compute $\left[G\right]$ using Eq (A2)
4. Generate a $M\times n$ matrix $\left[\overline{U}\right]$ using a standard normal distribution, where M and n are the number of simulations and number of demand parameters, respectively.
5. Compute the correlation and the square root of the variance matrix of the column vectors in $\left[G\right]$ and denote them as $R\left(\widehat{G}\right)$ and $\left[\sigma_G\right]$, respectively.
6. Using spectral value decomposition, $R\left(\widehat{G}\right)=\left[A_{Gcor}\right]\left[\lambda_{Gcor}\right]\left[A_{Gcor}\right]^T$
7. If OPT=1, then $\left[\overline{G}\right]=\left[\overline{U}\right]\left[\lambda_{Gcor}\right]^{1/2}\left[A_{Gcor}\right]^T\left[\sigma_G\right]$.
   If OPT=2, then
   i) subtract the sample mean from each column of $\left[\overline{U}\right]$ and calculate the covariance matrix $\Sigma\left(\widehat{\overline{U}}\right)$,
   ii) using spectral value decomposition, $\Sigma\left(\widehat{\overline{U}}\right)=\left[A_{\mathrm{cov}U}\right]\left[\lambda_{\mathrm{cov}U}\right]\left[A_{\mathrm{cov}U}\right]^T$
   iii) calculate $\left[\overline{G}\right]=\left[\overline{U}\right]\left[A_{\mathrm{cov}U}\right]\left[\lambda_{\mathrm{cov}U}\right]^{-1/2}\left[\lambda_{Gcor}\right]^{1/2}\left[A_{Gcor}\right]^T\left[\sigma_G\right]$
8. Compute $\left[\overline{Y}\right]$ by adding back the mean: $\overline{Y}_i = G_i + \mu_{Y_i}\left\{1\right\}$.
9. Generate demand vectors by taking the exponential of $\left[\overline{Y}\right]$.

MATLAB coding of the SVD algorithm

```
clear all
close all

%OPT=0;% no acceleration
OPT=1;% with acceleration

%Develop underlying statistics of the of the response history analysis
%step-1
%X=load('DP.txt');
[mrow,ncol]=size(X);

Msize=1000000;% simulation size

tic;
%step-2
Y=log(X);
mu=(mean(Y));
RY=corrcoef(Y);

%step-3
G=Y-ones(mrow,ncol)*diag(mu);

%step-4
colU=ncol;
U_star=randn(Msize,colU);

%step-5

RG=corrcoef(G);
Var_G=var(G);
Sigma_G=diag(sqrt(Var_G));

%step-6
[A_Gcor,LAM_Gcor,B_Gcor]=svd(RG);

    sqrt_LAM_Gcor=zeros(ncol,ncol);
    for j=1:ncol
        sqrt_LAM_Gcor(j,j)=sqrt(LAM_Gcor(j,j));
    end

%step-7
if OPT==0
   G_BAR=U_star*sqrt_LAM_Gcor*A_Gcor'*Sigma_G;

%  G_BAR=U_star*sqrt(LAM_Gcor)*A_Gcor'*Sigma_G;
else
    if OPT==1
  %step-7.1
    mu_U_star=mean(U_star);
    U_star=U_star-ones(Msize,colU)*diag(mu_U_star);
```

```
  %step-7.2
        covU=cov(U_star);
        [A_covU,LAM_covU,B_covU]=svd(covU);

       inv_sqrt_LAM_covU=zeros(ncol,ncol);
    for j=1:ncol
        inv_sqrt_LAM_covU(j,j)=1/sqrt(LAM_covU(j,j));
    end
  %step-7.3
     G_BAR=U_star*A_covU*inv_sqrt_LAM_covU*sqrt_LAM_Gcor*A_Gcor'*Sigma_G;

%    G_BAR=U_star*A_covU*inv(sqrt(LAM_covU))*sqrt(LAM_Gcor)*A_Gcor'*Sigma_G;
    end
end

%step-8
Y_BAR=G_BAR+ones(Msize,ncol)*diag(mu);

%step-9

W=exp(Y_BAR);

Time_Elapsed=toc;
save correlated_DP_svd_eff.txt W -ascii -double -tabs;

%Check results
mu_Y_BAR=mean(Y_BAR);
RY_BAR=corrcoef(Y_BAR);

G_8=mu_Y_BAR./mu %for Table G-8
G_9=RY_BAR./RY %for Table G-9
G_10=var(Y_BAR)./var(Y)
```

4. Evaluation of results generated by the PACT and DB algorithms

We evaluated the performance of the PACT (*cholcov*) and SVD algorithms using results of analysis of two sample structures: 1) a three-story building with seven demand parameters (3 story drifts and 4 floor accelerations), and 2) an eight-story building with seventeen demand parameters (8 story drifts and 9 floor accelerations). The base line study for the three-story building involved a maximum of 11 ground motions; 20 ground motions were used for analysis of the eight-story building.

Three-story building

Table 1 presents benchmark data that are used later for the purpose of comparing results. The matrix of data is denoted as X. We assume the demand parameters are jointly lognormal per the discussion above. The mean and variance for each demand parameter from matrix X are presented in Table 2. The mean and variance for each demand parameter computed for matrix $[Y]=\ln[X]$ are presented in Table 3. The correlation matrix, $[R_{YY}]$, computed for $[Y]$ is presented in Table 4.

Table 5 presents the results of computations using PACT (*cholcov* subroutine) and 100 simulations. The table presents the ratio of PACT to benchmark values of means (in the Y space) and correlation coefficients (in the Y space). The correlation coefficients are seen to vary significantly from the target values. We consider now three different sets of ground motions and the associated analysis results to investigate the impact of reducing *m* with respect to *n*: G1 to G7, G1 to G6 and G1 to G3. Results are presented in Table 6 through Table 8, respectively. The ratios are computed using 100 simulations and normalized by the values computed from the underlying demand parameter matrices. For all three cases, $m \leq n$ and the correlation matrix is rank deficient. The correlation coefficients in each case differ significantly from the target values because of the small number of simulations. This conclusion is verified by analysis using 10,000 and 1,000,000 simulations. Results are presented in Table 9 and Table 10, respectively and show that the statistical parameters converge to the target values as the number of simulations is increased, which is an expected result.

The performance of the SVD algorithm is considered next, initially without accelerating the convergence of results. We present these data to demonstrate the robustness of the SVD algorithm and use the *cholcov* data as a benchmark. Table 11 through Table 13 presents the results for $m = 11$ and 100, 10,000 and 100,0000 simulations, respectively. Compare these results with those of Tables 5, 9 and 10, respectively. The accuracy of the SVD results is comparable to that achieved with the *cholcov* algorithm. A similar trend is observed when the analysis is performed with other three combinations of ground motions, namely, G1 to G7, G1 to G6 and G1 to G3. The results for these cases are presented in Table 14 through Table 22. Similar to the *cholcov* algorithm, SVD algorithm also works for a rank-deficient covariance matrix. The SVD algorithm can also be used for a negative definite covariance matrix by assigning zero to those spectral values for which the left and right spectral shapes differ only by sign. The *cholcov* algorithm will not work in such a case as it requires the covariance matrix to be necessarily positive semi-definite (see the Matlab 8 documentation).

We now consider the results of analysis using the accelerated SVD algorithm. We repeated the analysis above for all combinations of ground motions and 100, 10,000 and 1,000,000 simulations. Table 23 presents results. For every analysis, the means and correlation coefficients are equal to the target values, regardless of the number of simulations.

Eight-story building

Analysis of the eight-story building was performed using demand parameters for hazard level 7 and direction 1. Benchmark data are provided for 20 ground motions and 17 demand parameters. Table 24 presents the demand parameter matrix in the X space. The mean and variance of the demand matrix are presented in Table 25 (X space) and Table 26 (Y space). Table 27 presents the correlation coefficient matrix in the Y space. Table 28 presents results in the same format as used previously for the *cholcov* algorithm and 100 simulations. The CPU time is also noted to enable a discussion on the relationship between execution time, number of simulations (accuracy) and choice of algorithm. Additional testing of the *cholcov* algorithm is performed with 10,000 and 1,000,000 simulations. Results are presented in Table 29 and Table 30, respectively. The CPU time and the precision increases with an increasing number of simulations.

Results computed using the SVD algorithm *without acceleration* for 100, 10,000 and 1,000,000 simulations are presented in Table 31 through Table 33, respectively. For a given number of simulations, the accuracy of the results is comparable to those generated using the *cholcov* algorithm. The computation times in SVD algorithm are slightly greater than that of *cholcov*. However, when acceleration option in the SVD algorithm is selected (see Table 34), the computational expense using the SVD algorithm is substantially less than *cholcov* for the same accuracy. (We have not considered here the minimum number of simulations required to generate a stable distribution of loss.)

The SVD algorithm can factorize matrices that are negative definite. The recovery of the underlying statistics can be accelerated with the SVD algorithm, which will be a computationally attractive feature when the number of simulations required to generate a stable loss curve is small and/or the number of demand parameters is large.

References

Mathworks, 2010, *MATLAB 8—The Language of Technical Computing*, Natick, MA

Wilkinson, J. H., 1965, *The Algebraic Eigenvalue Problem*, Oxford Science Publication

Yang, T.Y., Moehle, J., Stojadinovic, B., Der Kiureghian, A., 2006, "An application of PEER performance-based earthquake engineering methodology," *Proceedings*, 8th US National Conference on Earthquake Engineering, San Francisco, California.

Yang, T. Y., Moehle, J., Stojadinovic, B., and Der Kiureghian, A. 2009, "Performance evaluation of structural systems: theory and implementation," *Journal of Structural Engineering*, Vol. 135, No. 10, pg. 1146-1154.

Zareian, F. 2010. *Personal communication*

Table 1. Demand matrix, X , from response history analysis, example 1

	$\delta_1(\%)$	$\delta_2(\%)$	$\delta_3(\%)$	$a_1(g)$	$a_2(g)$	$a_3(g)$	$a_4(g)$
G1	1.26	1.45	1.71	0.54	0.87	0.88	0.65
G2	1.41	2.05	2.43	0.55	0.87	0.77	0.78
G3	1.37	1.96	2.63	0.75	1.04	0.89	0.81
G4	0.97	1.87	2.74	0.55	0.92	1.12	0.75
G5	0.94	1.8	2.02	0.40	0.77	0.74	0.64
G6	1.73	2.55	2.46	0.45	0.57	0.45	0.59
G7	1.05	2.15	2.26	0.38	0.59	0.49	0.52
G8	1.40	1.67	2.1	0.73	1.50	1.34	0.83
G9	1.59	1.76	2.01	0.59	0.94	0.81	0.72
G10	0.83	1.68	2.25	0.53	1.00	0.9	0.74
G11	0.96	1.83	2.25	0.49	0.90	0.81	0.64

Table 2. Demand matrix X statistics, example 1

	$\delta_1(\%)$	$\delta_2(\%)$	$\delta_3(\%)$	$a_1(g)$	$a_2(g)$	$a_3(g)$	$a_4(g)$
μ_x	1.2282	1.8882	2.2600	0.5418	0.9064	0.8364	0.6973
σ_x^2	0.0878	0.0849	0.0885	0.0139	0.0615	0.0627	0.0094

Table 3. Demand matrix Y statistics, example 1

	$\delta_1(\%)$	$\delta_2(\%)$	$\delta_3(\%)$	$a_1(g)$	$a_2(g)$	$a_3(g)$	$a_4(g)$
μ_x	0.179	0.625	0.807	-0.634	-0.131	-0.222	-0.370
σ_x^2	0.059	0.022	0.018	0.046	0.070	0.0993	0.021

Table 4. Benchmark correlation matrix, $[R_{YY}]$, example 1

1.000	0.339	-0.019	0.375	-0.022	-0.193	0.145
0.339	1.000	0.656	-0.353	-0.646	-0.723	-0.376
-0.019	0.656	1.000	0.136	-0.094	-0.066	0.220
0.375	-0.353	0.136	1.000	0.839	0.731	0.881
-0.022	-0.646	-0.094	0.839	1.000	0.934	0.863
-0.193	-0.723	-0.066	0.731	0.934	1.000	0.820
0.145	-0.376	0.220	0.881	0.863	0.820	1.000

Table 5. PACT algorithm using 11 ground motions, 100 simulations, example 1

Demand parameters						
δ_1	δ_2	δ_3	a_1	a_2	a_3	a_4
Ratio of means						
1.0347	1.0054	1.0010	1.0032	1.0215	1.0410	1.0004
Ratio of correlation coefficients						
1.0000	1.2520	0.6032	0.7389	5.3804	1.4156	0.4426
1.2520	1.0000	0.9645	0.8050	0.9612	1.0051	0.9401
0.6032	0.9645	1.0000	1.7816	0.2477	0.7994	1.2546
0.7389	0.8050	1.7816	1.0000	1.0010	0.9940	1.0031
5.3804	0.9612	0.2477	1.0010	1.0000	1.0129	0.9842
1.4156	1.0051	0.7994	0.9940	1.0129	1.0000	0.9888
0.4426	0.9401	1.2546	1.0031	0.9842	0.9888	1.0000

Table 6. PACT algorithm using 7 ground motions, 100 simulations, example 1

Demand parameters						
δ_1	δ_2	δ_3	a_1	a_2	a_3	a_4
Ratio of means						
0.9906	0.9949	0.9972	0.9933	0.9730	0.9796	0.9954
Ratio of correlation coefficients						
1.0000	0.9439	0.4691	0.8072	1.1266	1.0514	1.0087
0.9439	1.0000	1.0293	0.5630	0.8237	0.9044	0.4680
0.4691	1.0293	1.0000	1.2464	1.7927	1.8024	1.1625
0.8072	0.5630	1.2464	1.0000	1.0017	0.9681	0.9712
1.1266	0.8237	1.7927	1.0017	1.0000	0.9832	0.9823
1.0514	0.9044	1.8024	0.9681	0.9832	1.0000	0.9534
1.0087	0.4680	1.1625	0.9712	0.9823	0.9534	1.0000

Table 7. PACT algorithm using 6 ground motions, 100 simulations, example 1

Demand parameters						
δ_1	δ_2	δ_3	a_1	a_2	a_3	a_4
Ratio of means						
1.0012	0.9922	0.9983	0.9887	0.9642	0.9407	0.9945
Ratio of correlation coefficients						
1.0000	0.9590	1.6589	1.4868	0.6698	0.8480	-0.0301
0.9590	1.0000	1.0860	-0.1114	0.8210	0.8957	0.0960
1.6589	1.0860	1.0000	1.2653	1.3122	1.2228	1.0519
1.4868	-0.1114	1.2653	1.0000	1.0195	0.9924	1.0527
0.6698	0.8210	1.3122	1.0195	1.0000	0.9692	1.0063
0.8480	0.8957	1.2228	0.9924	0.9692	1.0000	0.9705
-0.0301	0.0960	1.0519	1.0527	1.0063	0.9705	1.0000

Table 8. PACT algorithm using 3 ground motions, 100 simulations, example 1

Demand parameters						
δ_1	δ_2	δ_3	a_1	a_2	a_3	a_4
Ratio of means						
0.9982	0.9901	0.9885	1.0086	1.0161	0.9986	1.0081
Ratio of correlation coefficients						
1.0000	1.0010	1.0183	1.4873	1.5998	0.8824	1.0170
1.0010	1.0000	1.0099	1.3168	1.3732	0.8603	1.0090
1.0183	1.0099	1.0000	1.1286	1.1479	0.8145	1.0000
1.4873	1.3168	1.1286	1.0000	1.0002	0.8754	1.1338
1.5998	1.3732	1.1479	1.0002	1.0000	0.8808	1.1539
0.8824	0.8603	0.8145	0.8754	0.8808	1.0000	0.8166
1.0170	1.0090	1.0000	1.1338	1.1539	0.8166	1.0000

Table 9. PACT algorithm using 11 ground motions, 10,000 simulations, example 1

Demand parameters						
δ_1	δ_2	δ_3	a_1	a_2	a_3	a_4
Ratio of means						
1.0006	1.0002	1.0000	1.0001	0.9992	1.0020	1.0003
Ratio of correlation coefficients						
1.0000	0.9867	1.8920	0.9932	1.3619	1.0442	0.8935
0.9867	1.0000	0.9940	1.0071	1.0001	0.9991	1.0019
1.8920	0.9940	1.0000	0.9588	0.9404	0.9025	1.0175
0.9932	1.0071	0.9588	1.0000	0.9978	0.9944	0.9947
1.3619	1.0001	0.9404	0.9978	1.0000	0.9995	1.0019
1.0442	0.9991	0.9025	0.9944	0.9995	1.0000	1.0009
0.8935	1.0019	1.0175	0.9947	1.0019	1.0009	1.0000

Table 10. PACT algorithm using 11 ground motions, 1,000,000 simulations, example 1

Demand parameters						
δ_1	δ_2	δ_3	a_1	a_2	a_3	a_4
Ratio of means						
1.0003	1.0000	0.9999	1.0000	1.0003	1.0003	1.0001
Ratio of correlation coefficients						
1.0000	1.0008	1.0599	0.9985	0.9982	1.0028	0.9979
1.0008	1.0000	0.9990	1.0001	0.9997	0.9992	0.9997
1.0599	0.9990	1.0000	0.9997	0.9941	0.9776	1.0013
0.9985	1.0001	0.9997	1.0000	1.0003	1.0001	1.0000
0.9982	0.9997	0.9941	1.0003	1.0000	0.9999	1.0002
1.0028	0.9992	0.9776	1.0001	0.9999	1.0000	1.0002
0.9979	0.9997	1.0013	1.0000	1.0002	1.0002	1.0000

Table 11. SVD algorithm using 11 ground motions, 100 simulations, example 1, no acceleration

Demand parameters						
δ_1	δ_2	δ_3	a_1	a_2	a_3	a_4
Ratio of means						
0.9910	0.9897	0.9759	1.0313	1.0558	1.0311	1.0452
Ratio of correlation coefficients						
1.0000	0.9031	1.3382	1.0446	2.1557	0.8650	1.1866
0.9031	1.0000	1.0596	0.8851	0.8649	0.9325	0.7957
1.3382	1.0596	1.0000	1.0812	0.0676	0.6748	1.1370
1.0446	0.8851	1.0812	1.0000	0.9865	0.9892	0.9771
2.1557	0.8649	0.0676	0.9865	1.0000	0.9857	0.9805
0.8650	0.9325	0.6748	0.9892	0.9857	1.0000	0.9916
1.1866	0.7957	1.1370	0.9771	0.9805	0.9916	1.0000

Table 12. SVD algorithm using 11 ground motions, 10,000 simulations, example 1, no acceleration

Demand parameters						
δ_1	δ_2	δ_3	a_1	a_2	a_3	a_4
Ratio of means						
0.9905	1.0034	1.0016	1.0075	1.0455	1.0198	1.0061
Ratio of correlation coefficients						
1.0000	0.9660	1.4111	1.0198	0.9692	0.9909	0.9896
0.9660	1.0000	1.0009	1.0027	1.0057	1.0027	1.0112
1.4111	1.0009	1.0000	0.9839	0.9986	0.9976	0.9995
1.0198	1.0027	0.9839	1.0000	0.9972	0.9952	0.9978
0.9692	1.0057	0.9986	0.9972	1.0000	1.0004	0.9990
0.9909	1.0027	0.9976	0.9952	1.0004	1.0000	1.0003
0.9896	1.0112	0.9995	0.9978	0.9990	1.0003	1.0000

Table 13. SVD algorithm using 11 ground motions, 1,000,000 simulations, example 1, no acceleration

Demand parameters						
δ_1	δ_2	δ_3	a_1	a_2	a_3	a_4
Ratio of means						
1.0010	0.9999	0.9996	1.0003	1.0017	1.0016	1.0006
Ratio of correlation coefficients						
1.0000	1.0006	1.0533	0.9964	1.0520	1.0049	0.9849
1.0006	1.0000	0.9985	1.0002	1.0000	1.0001	1.0016
1.0533	0.9985	1.0000	1.0078	0.9893	0.9849	1.0032
0.9964	1.0002	1.0078	1.0000	0.9999	1.0000	0.9996
1.0520	1.0000	0.9893	0.9999	1.0000	0.9999	1.0002
1.0049	1.0001	0.9849	1.0000	0.9999	1.0000	1.0004
0.9849	1.0016	1.0032	0.9996	1.0002	1.0004	1.0000

Table 14. SVD algorithm using 7 ground motions, 100 simulations, example 1, no acceleration

Demand parameters						
δ_1	δ_2	δ_3	a_1	a_2	a_3	a_4
Ratio of means						
1.0494	1.0059	0.9815	1.0310	1.0792	1.1072	1.0291
Ratio of correlation coefficients						
1.0000	1.1490	0.9732	0.7334	1.8607	1.2955	-0.8391
1.1490	1.0000	0.8902	1.2063	1.1220	1.0901	1.5519
0.9732	0.8902	1.0000	1.0843	1.0665	1.0783	0.9336
0.7334	1.2063	1.0843	1.0000	0.9718	0.9803	0.9698
1.8607	1.1220	1.0665	0.9718	1.0000	1.0125	1.0235
1.2955	1.0901	1.0783	0.9803	1.0125	1.0000	1.0507
-0.8391	1.5519	0.9336	0.9698	1.0235	1.0507	1.0000

Table 15. SVD algorithm using 7 ground motions, 10,000 simulations, example 1, no acceleration

Demand parameters						
δ_1	δ_2	δ_3	a_1	a_2	a_3	a_4
Ratio of means						
0.9795	0.9977	0.9989	1.0025	0.9994	0.9947	1.0022
Ratio of correlation coefficients						
1.0000	0.9804	0.9146	0.9880	0.9890	0.9859	0.9849
0.9804	1.0000	0.9919	1.0183	0.9916	0.9900	0.9773
0.9146	0.9919	1.0000	1.0097	1.1247	1.2533	1.0337
0.9880	1.0183	1.0097	1.0000	1.0046	1.0137	1.0021
0.9890	0.9916	1.1247	1.0046	1.0000	1.0022	1.0007
0.9859	0.9900	1.2533	1.0137	1.0022	1.0000	1.0045
0.9849	0.9773	1.0337	1.0021	1.0007	1.0045	1.0000

Table 16. SVD algorithm using 7 ground motions, 1,000,000 simulations, example 1, no acceleration

Demand parameters						
δ_1	δ_2	δ_3	a_1	a_2	a_3	a_4
Ratio of means						
1.0006	1.0000	0.9999	1.0001	1.0000	1.0004	0.9999
Ratio of correlation coefficients						
1.0000	0.9982	0.9949	1.0032	0.9937	0.9974	1.0067
0.9982	1.0000	0.9989	1.0013	1.0002	1.0005	1.0032
0.9949	0.9989	1.0000	0.9999	1.0044	1.0069	0.9992
1.0032	1.0013	0.9999	1.0000	1.0000	0.9999	1.0000
0.9937	1.0002	1.0044	1.0000	1.0000	0.9999	1.0003
0.9974	1.0005	1.0069	0.9999	0.9999	1.0000	1.0002
1.0067	1.0032	0.9992	1.0000	1.0003	1.0002	1.0000

Table 17. SVD algorithm using 6 ground motions, 100 simulations, example 1, no acceleration

Demand parameters						
δ_1	δ_2	δ_3	a_1	a_2	a_3	a_4
Ratio of means						
1.0377	1.0029	1.0112	0.9722	0.9414	0.9550	0.9727
Ratio of correlation coefficients						
1.0000	1.0181	0.8764	0.5729	1.2440	1.0961	1.8301
1.0181	1.0000	1.0534	1.0365	1.0388	0.9806	1.0578
0.8764	1.0534	1.0000	0.9088	0.3790	0.3916	0.9284
0.5729	1.0365	0.9088	1.0000	0.9738	1.0122	0.9757
1.2440	1.0388	0.3790	0.9738	1.0000	1.0106	0.9909
1.0961	0.9806	0.3916	1.0122	1.0106	1.0000	1.0246
1.8301	1.0578	0.9284	0.9757	0.9909	1.0246	1.0000

Table 18. SVD algorithm using 6 ground motions, 10,000 simulations, example 1, no acceleration

Demand parameters						
δ_1	δ_2	δ_3	a_1	a_2	a_3	a_4
Ratio of means						
1.0153	1.0000	0.9981	0.9995	1.0054	1.0111	1.0017
Ratio of correlation coefficients						
1.0000	1.0053	1.0239	0.9429	1.0176	1.0077	1.0181
1.0053	1.0000	1.0090	1.0439	1.0101	1.0032	1.0148
1.0239	1.0090	1.0000	0.9783	0.8646	0.7859	0.9838
0.9429	1.0439	0.9783	1.0000	1.0030	1.0082	1.0054
1.0176	1.0101	0.8646	1.0030	1.0000	1.0001	0.9971
1.0077	1.0032	0.7859	1.0082	1.0001	1.0000	0.9967
1.0181	1.0148	0.9838	1.0054	0.9971	0.9967	1.0000

Table 19. SVD algorithm using 6 ground motions, 1,000,000 simulations, example 1, no acceleration

Demand parameters						
δ_1	δ_2	δ_3	a_1	a_2	a_3	a_4
Ratio of means						
0.9995	1.0001	1.0000	1.0005	1.0014	1.0004	1.0005
Ratio of correlation coefficients						
1.0000	0.9993	1.0051	1.0075	0.9968	0.9984	0.9905
0.9993	1.0000	1.0001	1.0006	1.0000	1.0003	1.0005
1.0051	1.0001	1.0000	0.9996	0.9984	0.9915	0.9995
1.0075	1.0006	0.9996	1.0000	0.9999	0.9997	0.9998
0.9968	1.0000	0.9984	0.9999	1.0000	0.9999	1.0000
0.9984	1.0003	0.9915	0.9997	0.9999	1.0000	0.9997
0.9905	1.0005	0.9995	0.9998	1.0000	0.9997	1.0000

Table 20. SVD algorithm using 3 ground motions, 100 simulations, example 1, no acceleration

Demand parameters						
δ_1	δ_2	δ_3	a_1	a_2	a_3	a_4
Ratio of means						
1.0198	1.0319	1.0261	0.9889	0.9674	1.0318	0.9638
Ratio of correlation coefficients						
1.0000	1.0013	1.0089	0.8330	0.7768	1.1076	1.0086
1.0013	1.0000	1.0030	0.8629	0.8277	1.1572	1.0028
1.0089	1.0030	1.0000	0.9152	0.8981	1.3930	1.0000
0.8330	0.8629	0.9152	1.0000	0.9996	0.9343	0.9132
0.7768	0.8277	0.8981	0.9996	1.0000	0.9512	0.8955
1.1076	1.1572	1.3930	0.9343	0.9512	1.0000	1.3752
1.0086	1.0028	1.0000	0.9132	0.8955	1.3752	1.0000

Table 21. SVD algorithm using 3 ground motions, 10,000 simulations, example 1, no acceleration

Demand parameters						
δ_1	δ_2	δ_3	a_1	a_2	a_3	a_4
Ratio of means						
1.0028	1.0044	1.0034	0.9993	0.9984	1.0054	0.9953
Ratio of correlation coefficients						
1.0000	0.9999	0.9990	1.0150	1.0203	0.9879	0.9990
0.9999	1.0000	0.9997	1.0127	1.0160	0.9824	0.9997
0.9990	0.9997	1.0000	1.0078	1.0094	0.9568	1.0000
1.0150	1.0127	1.0078	1.0000	1.0000	1.0085	1.0080
1.0203	1.0160	1.0094	1.0000	1.0000	1.0066	1.0097
0.9879	0.9824	0.9568	1.0085	1.0066	1.0000	0.9587
0.9990	0.9997	1.0000	1.0080	1.0097	0.9587	1.0000

Table 22. SVD algorithm using 3 ground motions, 1,000,000 simulations, example 1, no acceleration

Demand parameters						
δ_1	δ_2	δ_3	a_1	a_2	a_3	a_4
Ratio of means						
1.0004	1.0007	1.0008	0.9990	0.9967	0.9998	0.9990
Ratio of correlation coefficients						
1.0000	1.0000	1.0000	1.0015	1.0019	0.9995	1.0000
1.0000	1.0000	1.0000	1.0011	1.0013	0.9994	1.0000
1.0000	1.0000	1.0000	1.0005	1.0006	0.9988	1.0000
1.0015	1.0011	1.0005	1.0000	1.0000	0.9998	1.0005
1.0019	1.0013	1.0006	1.0000	1.0000	0.9998	1.0006
0.9995	0.9994	0.9988	0.9998	0.9998	1.0000	0.9988
1.0000	1.0000	1.0000	1.0005	1.0006	0.9988	1.0000

Table 23. SVD algorithm, example 1, with acceleration

Demand parameters						
δ_1	δ_2	δ_3	a_1	a_2	a_3	a_4
Ratio of means						
1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
Ratio of correlation coefficients						
1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table 24. Demand matrix, X, from response history analysis, example 2

Floor acceleration $a_i(g)$								Drift ratio $\delta_i(\%)$							
2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8
9.75 E-02	1.21 E-01	9.79 E-02	1.18 E-01	1.10 E-01	1.22 E-01	1.24 E-01	2.78 E-01	8.82 E-02	9.41 E-02	9.78 E-02	8.55 E-02	8.43 E-02	1.05 E-01	1.14 E-01	9.47 E-02
3.62 E-01	3.87 E-01	6.56 E-02	9.19 E-02	8.76 E-02	8.05 E-02	8.61 E-02	1.66 E-01	6.72 E-02	8.62 E-02	6.66 E-02	6.08 E-02	6.77 E-02	7.69 E-02	7.61 E-02	7.22 E-02
3.95 E-01	4.48 E-01	5.22 E-02	1.80 E-01	7.20 E-02	7.94 E-02	1.71 E-01	2.49 E-01	6.64 E-02	1.03 E-01	6.36 E-02	4.11 E-01	6.26 E-02	1.57 E-01	9.17 E-02	7.54 E-02
4.10 E-01	5.77 E+00	5.69 E-02	2.60 E-01	7.61 E-02	8.59 E-02	2.63 E-01	2.29 E-01	7.65 E-02	1.80 E-01	7.52 E-02	3.02 E+00	6.08 E-02	2.52 E-01	8.54 E-02	7.02 E-02
1.00 E+00	1.42 E+01	7.79 E-02	5.30 E-01	7.79 E-02	7.54 E-02	5.43 E-01	1.75 E-01	7.35 E-02	4.78 E-01	6.49 E-02	5.55 E+00	6.38 E-02	5.47 E-01	7.64 E-02	7.64 E-02
1.09 E+01	2.05 E+00	7.55 E-02	1.12 E+00	7.91 E-02	8.30 E-02	1.16 E+00	2.14 E-01	6.55 E-02	3.55 E-01	4.91 E-02	8.66 E-01	7.59 E-02	1.15 E+00	8.01 E-02	7.28 E-02
2.21 E+01	2.05 E+01	5.88 E-02	1.25 E+00	7.45 E-02	7.92 E-02	1.29 E+00	1.96 E-01	6.90 E-02	4.89 E-01	5.49 E-02	6.24 E+00	6.45 E-02	1.28 E+00	8.44 E-02	8.94 E-02
1.36 E+01	1.24 E+01	4.29 E-02	1.30 E+00	7.90 E-02	6.12 E-02	3.69 E-01	2.19 E-01	5.05 E-02	4.14 E-01	4.46 E-02	3.56 E+00	5.59 E-02	3.63 E-01	7.18 E-02	1.05 E-01
5.60 E-02	6.30 E-02	3.36 E-02	6.75 E-02	5.15 E-02	3.53 E-02	1.59 E-01	1.06 E-01	1.57 E-01	2.52 E-02	5.65 E-02	7.16 E-02	1.19 E-01	4.52 E-02	1.12 E-01	9.85 E-02
6.54 E-03	8.34 E-03	6.92 E-03	7.69 E-03	7.65 E-03	8.37 E-03	8.80 E-03	1.09 E-02	6.58 E-03	7.13 E-03	6.85 E-03	6.14 E-03	5.79 E-03	7.78 E-03	6.62 E-03	6.31 E-03
1.00 E-02	1.27 E-02	1.01 E-02	1.22 E-02	1.38 E-02	1.22 E-02	1.55 E-02	1.13 E-02	1.02 E-02	1.11 E-02	1.04 E-02	9.54 E-03	8.79 E-03	1.29 E-02	1.05 E-02	9.46 E-03
1.98 E-02	1.84 E-02	1.26 E-02	1.85 E-02	1.87 E-02	1.59 E-02	2.19 E-02	1.35 E-02	1.25 E-02	1.55 E-02	1.30 E-02	1.67 E-02	1.23 E-02	1.87 E-02	1.46 E-02	1.26 E-02
3.29 E-02	2.66 E-02	1.22 E-02	3.35 E-02	2.14 E-02	2.02 E-02	3.03 E-02	1.60 E-02	1.33 E-02	1.99 E-02	1.29 E-02	2.75 E-02	1.60 E-02	3.38 E-02	1.50 E-02	1.75 E-02
3.57 E-02	2.99 E-02	1.40 E-02	4.55 E-02	2.68 E-02	2.53 E-02	4.52 E-02	1.72 E-02	1.55 E-02	2.27 E-02	1.53 E-02	2.93 E-02	1.77 E-02	4.79 E-02	1.66 E-02	2.06 E-02
1.68 E-01	1.71 E-01	1.78 E-02	8.83 E-02	3.65 E-02	3.95 E-02	9.99 E-02	1.85 E-02	1.78 E-02	1.42 E-01	1.80 E-02	1.73 E-01	1.85 E-02	9.51 E-02	2.04 E-02	2.12 E-02
1.87 E-01	1.87 E-01	2.08 E-02	2.54 E-01	4.22 E-02	4.70 E-02	2.74 E-01	1.77 E-02	1.76 E-02	1.60 E-01	1.66 E-02	1.91 E-01	1.83 E-02	2.67 E-01	2.03 E-02	2.04 E-02
1.87 E-01	1.84 E-01	1.77 E-02	2.56 E-01	3.95 E-02	4.57 E-02	2.74 E-01	1.29 E-02	1.29 E-02	1.62 E-01	1.08 E-02	1.88 E-01	1.31 E-02	2.68 E-01	1.50 E-02	1.40 E-02
9.75 E-02	1.21 E-01	9.79 E-02	1.18 E-01	1.10 E-01	1.22 E-01	1.24 E-01	2.78 E-01	8.82 E-02	9.41 E-02	9.78 E-02	8.55 E-02	8.43 E-02	1.05 E-01	1.14 E-01	9.47 E-02
3.62 E-01	3.87 E-01	6.56 E-02	9.19 E-02	8.76 E-02	8.05 E-02	8.61 E-02	1.66 E-01	6.72 E-02	8.62 E-02	6.66 E-02	6.08 E-02	6.77 E-02	7.69 E-02	7.61 E-02	7.22 E-02
3.95 E-01	4.48 E-01	5.22 E-02	1.80 E-01	7.20 E-02	7.94 E-02	1.71 E-01	2.49 E-01	6.64 E-02	1.03 E-01	6.36 E-02	4.11 E-01	6.26 E-02	1.57 E-01	9.17 E-02	7.54 E-02

Table 25. Demand matrix X statistics, example 2

	Floor acceleration $a_i(g)$						Drift ratio $\delta_i(\%)$							
	4	5	6	7	8	9	1	2	3	4	5	6	7	8
9	0.5642	1.1961	0.9372	2.7139	1.6523	0.0883	0.0075	0.0114	0.0158	0.0210	0.0253	0.0620	0.0917	0.0888
0	1.9247	10.8438	5.8087	42.3952	15.7385	0.0025	0.0000	0.0000	0.0000	0.0001	0.0001	0.0034	0.0097	0.0101

Table 26. Demand matrix Y statistics, example 2

	Floor Acceleration $a_i(g)$						Drift Ratio $\delta_i(\%)$							
	4	5	6	7	8	9	1	2	3	4	5	6	7	8
671	-1.8246	-1.5619	-1.5087	-1.2277	-1.5208	-2.5886	-4.9071	-4.4858	-4.1665	-3.9196	-3.7558	-3.1942	-3.0149	-3.1944
312	1.6073	2.5135	2.3479	3.9575	3.3440	0.3570	0.0239	0.0204	0.0324	0.1148	0.1593	0.8381	1.3408	1.7432

Table 27. Benchmark correlation matrix, $[R_{YY}]$, example 2

0.0898	0.0122	0.0606	-0.0042	0.0290	0.0181	0.8877	0.4275	0.1514	0.0502	0.0388	-0.1253	-0.0947	-0.0957
0.5937	0.5941	0.6923	0.7022	0.7472	-0.1372	0.3179	0.1659	0.4425	0.4284	0.3189	0.4924	0.3836	0.3806
0.9145	0.9040	0.8818	0.9308	0.9351	-0.1045	0.2464	0.1565	0.6192	0.7164	0.5943	0.7746	0.7032	0.6864
1.0000	0.9791	0.7617	0.8963	0.8921	-0.1443	0.1343	0.1450	0.5366	0.6228	0.5107	0.7882	0.6941	0.6798
0.9791	1.0000	0.8395	0.9422	0.9283	-0.2261	0.0999	0.1798	0.6100	0.7034	0.6021	0.8738	0.7973	0.7833
0.7617	0.8395	1.0000	0.9613	0.9330	-0.2061	0.1286	0.2373	0.7528	0.8420	0.7585	0.8711	0.8579	0.8425
0.8963	0.9422	0.9613	1.0000	0.9833	-0.1843	0.0614	0.1640	0.6920	0.7923	0.6866	0.8963	0.8342	0.8186
0.8921	0.9283	0.9330	0.9833	1.0000	-0.1949	0.0546	0.0994	0.6401	0.7584	0.6351	0.8698	0.7832	0.7695
-0.1443	-0.2261	-0.2061	-0.1843	-0.1949	1.0000	-0.1257	-0.1405	-0.2785	-0.2104	-0.1686	-0.3340	-0.2787	-0.2886
0.1343	0.0999	0.1286	0.0614	0.0546	-0.1257	1.0000	0.7365	0.3841	0.2202	0.2461	0.0464	0.1005	0.1024
0.1450	0.1798	0.2373	0.1640	0.0994	-0.1405	0.7365	1.0000	0.7235	0.4615	0.5377	0.2796	0.3906	0.3910
0.5366	0.6100	0.7528	0.6920	0.6401	-0.2785	0.3841	0.7235	1.0000	0.8795	0.8579	0.7546	0.7921	0.7840
0.6228	0.7034	0.8420	0.7923	0.7584	-0.2104	0.2202	0.4615	0.8795	1.0000	0.9738	0.8600	0.9036	0.8951
0.5107	0.6021	0.7585	0.6866	0.6351	-0.1686	0.2461	0.5377	0.8579	0.9738	1.0000	0.8057	0.8987	0.8947
0.7882	0.8738	0.8711	0.8963	0.8698	-0.3340	0.0464	0.2796	0.7546	0.8600	0.8057	1.0000	0.9588	0.9572
0.6941	0.7973	0.8579	0.8342	0.7832	-0.2787	0.1005	0.3906	0.7921	0.9036	0.8987	0.9588	1.0000	0.9979
0.6798	0.7833	0.8425	0.8186	0.7695	-0.2886	0.1024	0.3910	0.7840	0.8951	0.8947	0.9572	0.9979	1.0000

Table 28. PACT algorithm using 20 ground motions, 100 simulations, example 2

Demand parameters													
Floor acceleration $a_i(g)$						Drift ratio $\delta_i(\%)$							
4	5	6	7	8	9	1	2	3	4	5	6	7	8
Ratio of means													
0.8891	0.7839	0.8597	0.6309	0.7862	1.0128	1.0001	0.9997	0.9991	0.9972	0.9964	0.9806	0.9682	0.9680
Ratio of correlation coefficients													
1.3968	4.6137	1.4655	-4.1307	1.5317	-3.1156	1.0022	0.9925	0.8532	2.5022	3.4967	0.5472	0.0826	0.1441
1.0028	1.0048	1.0501	1.0359	1.0281	1.6025	0.5870	0.1366	0.9031	0.9440	0.8302	0.9711	0.9329	0.9269
0.9964	1.0080	1.0230	1.0084	1.0058	2.4458	0.9932	1.0307	1.0549	1.0185	0.9877	1.0296	1.0265	1.0273
1.0000	1.0052	1.0077	0.9986	1.0005	2.3642	1.1456	1.1572	1.1253	1.0664	1.0128	1.0506	1.0281	1.0291
1.0052	1.0000	0.9954	0.9940	0.9966	1.6825	1.0774	1.0125	1.0852	1.0470	0.9974	1.0252	1.0033	1.0040
1.0077	0.9954	1.0000	1.0011	0.9966	1.2326	0.6422	0.7285	0.9780	0.9640	0.9297	0.9742	0.9601	0.9599
0.9986	0.9940	1.0011	1.0000	0.9981	1.5201	0.4103	0.7656	1.0261	0.9818	0.9310	0.9921	0.9673	0.9675
1.0005	0.9966	0.9966	0.9981	1.0000	1.5689	0.1159	0.4879	1.0198	0.9797	0.9239	0.9881	0.9633	0.9637
2.3642	1.6825	1.2326	1.5201	1.5689	1.0000	1.2295	0.6549	0.9552	1.5986	1.7228	1.3185	1.4079	1.4054
1.1456	1.0774	0.6422	0.4103	0.1159	1.2295	1.0000	0.9930	0.8669	1.2238	1.2799	1.2068	1.3613	1.3026
1.1572	1.0125	0.7285	0.7656	0.4879	0.6549	0.9930	1.0000	0.9639	1.0764	1.0857	1.0842	1.0512	1.0499
1.1253	1.0852	0.9780	1.0261	1.0198	0.9552	0.8669	0.9639	1.0000	1.0061	0.9904	1.0584	1.0314	1.0339
1.0664	1.0470	0.9640	0.9818	0.9797	1.5986	1.2238	1.0764	1.0061	1.0000	0.9961	1.0193	1.0267	1.0280
1.0128	0.9974	0.9297	0.9310	0.9239	1.7228	1.2799	1.0857	0.9904	0.9961	1.0000	0.9893	1.0104	1.0111
1.0506	1.0252	0.9742	0.9921	0.9881	1.3185	1.2068	1.0842	1.0584	1.0193	0.9893	1.0000	0.9963	0.9959
1.0281	1.0033	0.9601	0.9673	0.9633	1.4079	1.3613	1.0512	1.0314	1.0267	1.0104	0.9963	1.0000	1.0001
1.0291	1.0040	0.9599	0.9675	0.9637	1.4054	1.3026	1.0499	1.0339	1.0280	1.0111	0.9959	1.0001	1.0000
CPU time = 0.0662 second													

Table 29. PACT algorithm using 20 ground motions, 10,000 simulations, example 2

Demand parameters													
Floor acceleration $a_i(g)$						Drift ratio $\delta_i(\%)$							
4	5	6	7	8	9	1	2	3	4	5	6	7	8
Ratio of means													
0.9892	0.9822	0.9785	0.9561	0.9644	1.0005	0.9999	0.9999	0.9999	0.9997	0.9997	0.9982	0.9973	0.9969
Ratio of correlation coefficients													
0.9175	0.6290	0.9747	2.0204	0.7793	0.5878	0.9993	1.0066	1.0096	0.9771	1.0178	1.0417	1.0008	1.0134
0.9980	1.0017	1.0011	1.0020	0.9996	1.0073	0.9630	0.9773	1.0050	1.0004	1.0057	1.0004	1.0074	1.0069
0.9976	1.0001	1.0016	1.0016	1.0009	1.0273	0.9872	1.0164	1.0057	1.0054	1.0117	1.0030	1.0088	1.0090
1.0000	1.0001	0.9924	0.9969	0.9979	1.0294	0.9761	0.9993	0.9971	0.9973	1.0017	1.0008	1.0020	1.0027
1.0001	1.0000	0.9948	0.9982	0.9995	1.0072	0.9942	1.0051	0.9993	0.9989	1.0021	1.0003	1.0012	1.0017
0.9924	0.9948	1.0000	0.9994	0.9995	0.9731	1.0015	1.0056	1.0017	1.0013	1.0041	0.9964	1.0006	1.0005
0.9969	0.9982	0.9994	1.0000	1.0003	0.9893	0.9791	1.0084	1.0027	1.0023	1.0060	1.0000	1.0032	1.0035
0.9979	0.9995	0.9995	1.0003	1.0000	0.9911	0.9601	1.0291	1.0051	1.0047	1.0098	1.0012	1.0054	1.0058
1.0294	1.0072	0.9731	0.9893	0.9911	1.0000	1.0741	1.0038	0.9808	0.9669	0.9569	0.9893	0.9772	0.9775
0.9761	0.9942	1.0015	0.9791	0.9601	1.0741	1.0000	1.0028	1.0089	1.0013	1.0041	0.9653	1.0177	1.0065
0.9993	1.0051	1.0056	1.0084	1.0291	1.0038	1.0028	1.0000	1.0023	0.9998	1.0001	0.9984	1.0014	1.0000
0.9971	0.9993	1.0017	1.0027	1.0051	0.9808	1.0089	1.0023	1.0000	1.0000	1.0021	0.9988	1.0015	1.0012
0.9973	0.9989	1.0013	1.0023	1.0047	0.9669	1.0013	0.9998	1.0000	1.0000	1.0007	0.9980	1.0002	1.0000
1.0017	1.0021	1.0041	1.0060	1.0098	0.9569	1.0041	1.0001	1.0021	1.0007	1.0000	0.9994	1.0003	0.9999
1.0008	1.0003	0.9964	1.0000	1.0012	0.9893	0.9653	0.9984	0.9988	0.9980	0.9994	1.0000	1.0000	1.0002
1.0020	1.0012	1.0006	1.0032	1.0054	0.9772	1.0177	1.0014	1.0015	1.0002	1.0003	1.0000	1.0000	1.0000
1.0027	1.0017	1.0005	1.0035	1.0058	0.9775	1.0065	1.0000	1.0012	1.0000	0.9999	1.0002	1.0000	1.0000
CPU time = 0.0941 second													

Table 30. PACT algorithm using 20 ground motions, 1,000,000 simulations, example 2

Demand parameters													
Floor acceleration $a_i(g)$						Drift ratio $\delta_i(\%)$							
4	5	6	7	8	9	1	2	3	4	5	6	7	8
Ratio of means													
0.9995	0.9994	1.0009	1.0004	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0002	1.0002
Ratio of correlation coefficients													
0.9814	0.8512	0.9654	1.4980	0.9255	0.9666	0.9998	0.9980	0.9862	0.9715	0.9743	1.0146	1.0150	1.0149
1.0010	1.0009	1.0003	1.0004	1.0005	0.9973	0.9924	0.9922	0.9986	0.9984	0.9972	1.0006	1.0003	1.0003
1.0003	1.0002	1.0002	1.0000	0.9999	0.9999	0.9938	1.0001	1.0007	1.0000	0.9998	1.0002	1.0002	1.0001
1.0000	1.0001	1.0012	1.0004	1.0002	1.0010	0.9909	1.0031	1.0020	1.0005	1.0004	1.0003	1.0006	1.0005
1.0001	1.0000	1.0007	1.0002	1.0001	0.9987	0.9856	1.0012	1.0012	1.0001	0.9999	1.0001	1.0002	1.0001
1.0012	1.0007	1.0000	1.0001	1.0002	0.9969	0.9859	0.9987	1.0001	0.9996	0.9993	1.0003	1.0001	1.0001
1.0004	1.0002	1.0001	1.0000	1.0000	0.9972	0.9707	0.9992	1.0004	0.9995	0.9992	1.0001	0.9999	0.9999
1.0002	1.0001	1.0002	1.0000	1.0000	0.9983	0.9643	0.9959	1.0002	0.9994	0.9989	0.9999	0.9998	0.9997
1.0010	0.9987	0.9969	0.9972	0.9983	1.0000	1.0041	1.0028	0.9993	0.9981	0.9976	0.9984	0.9987	0.9990
0.9909	0.9856	0.9859	0.9707	0.9643	1.0041	1.0000	0.9994	0.9951	0.9945	0.9972	0.9700	0.9906	0.9911
1.0031	1.0012	0.9987	0.9992	0.9959	1.0028	0.9994	1.0000	0.9994	1.0002	1.0008	1.0011	1.0014	1.0015
1.0020	1.0012	1.0001	1.0004	1.0002	0.9993	0.9951	0.9994	1.0000	1.0002	1.0002	1.0009	1.0008	1.0009
1.0005	1.0001	0.9996	0.9995	0.9994	0.9981	0.9945	1.0002	1.0002	1.0000	1.0000	1.0001	1.0001	1.0001
1.0004	0.9999	0.9993	0.9992	0.9989	0.9976	0.9972	1.0008	1.0002	1.0000	1.0000	0.9999	1.0000	1.0000
1.0003	1.0001	1.0003	1.0001	0.9999	0.9984	0.9700	1.0011	1.0009	1.0001	0.9999	1.0000	1.0000	1.0000
1.0006	1.0002	1.0001	0.9999	0.9998	0.9987	0.9906	1.0014	1.0008	1.0001	1.0000	1.0000	1.0000	1.0000
1.0005	1.0001	1.0001	0.9999	0.9997	0.9990	0.9911	1.0015	1.0009	1.0001	1.0000	1.0000	1.0000	1.0000
CPU time = 2.7443 second													

Table 31. SVD algorithm using 20 ground motions, 100 simulations, example 2, no acceleration

Demand Parameters													
Floor Acceleration $a_i(g)$						Drift Ratio $\delta_i(\%)$							
4	5	6	7	8	9	1	2	3	4	5	6	7	8
Ratio of means													
1.1174	1.1870	1.1423	1.2668	1.2058	0.9535	1.0007	1.0023	1.0070	1.0091	1.0056	1.0436	1.0385	1.0396
Ratio of correlation coefficients													
1.5607	4.1887	1.4149	-5.7751	1.2755	6.2742	0.9894	0.9215	0.9145	2.6206	3.1913	0.8307	0.5830	0.6735
1.0475	1.0802	1.0727	1.0721	1.0563	1.1439	0.9497	1.0528	1.0550	1.2477	1.3481	1.1829	1.2502	1.2544
1.0051	0.9941	0.9816	0.9878	0.9885	0.2658	1.2380	1.5456	1.0459	1.0149	1.0169	0.9802	0.9780	0.9741
1.0000	0.9955	0.9865	0.9891	0.9958	0.6539	1.3209	1.2722	1.0028	0.9673	0.9520	0.9779	0.9625	0.9567
0.9955	1.0000	0.9976	0.9973	1.0009	0.9863	1.5934	1.4872	1.0516	0.9961	1.0008	1.0008	0.9956	0.9950
0.9865	0.9976	1.0000	1.0013	0.9986	1.1353	1.6869	1.6444	1.0892	1.0488	1.0676	1.0209	1.0265	1.0309
0.9891	0.9973	1.0013	1.0000	1.0010	1.0593	2.1831	1.7485	1.0764	1.0266	1.0392	1.0133	1.0146	1.0169
0.9958	1.0009	0.9986	1.0010	1.0000	0.9096	1.7536	1.9808	1.0668	1.0235	1.0390	1.0141	1.0179	1.0207
0.6539	0.9863	1.1353	1.0593	0.9096	1.0000	1.0740	1.9675	1.2126	1.3349	1.7177	1.1959	1.3642	1.3647
1.3209	1.5934	1.6869	2.1831	1.7536	1.0740	1.0000	0.9823	1.0188	1.5076	1.5038	2.5276	1.9475	1.8526
1.2722	1.4872	1.6444	1.7485	1.9808	1.9675	0.9823	1.0000	1.0521	1.3221	1.2850	1.5279	1.4193	1.4117
1.0028	1.0516	1.0892	1.0764	1.0668	1.2126	1.0188	1.0521	1.0000	1.0584	1.0637	1.0884	1.0927	1.0944
0.9673	0.9961	1.0488	1.0266	1.0235	1.3349	1.5076	1.3221	1.0584	1.0000	1.0024	1.0046	1.0092	1.0095
0.9520	1.0008	1.0676	1.0392	1.0390	1.7177	1.5038	1.2850	1.0637	1.0024	1.0000	1.0095	1.0109	1.0099
0.9779	1.0008	1.0209	1.0133	1.0141	1.1959	2.5276	1.5279	1.0884	1.0046	1.0095	1.0000	1.0011	1.0015
0.9625	0.9956	1.0265	1.0146	1.0179	1.3642	1.9475	1.4193	1.0927	1.0092	1.0109	1.0011	1.0000	1.0004
0.9567	0.9950	1.0309	1.0169	1.0207	1.3647	1.8526	1.4117	1.0944	1.0095	1.0099	1.0015	1.0004	1.0000
CPU time = 0.0662 second													

Table 32. SVD algorithm using 20 ground motions, 10,000 simulations, example 2, no acceleration

Demand Parameters													
Floor Acceleration $a_i(g)$						Drift Ratio $\delta_i(\%)$							
4	5	6	7	8	9	1	2	3	4	5	6	7	8
Ratio of means													
1.0031	1.0044	0.9998	1.0032	1.0044	1.0001	0.9999	0.9998	0.9997	1.0001	1.0000	1.0020	1.0017	1.0020
Ratio of correlation coefficients													
0.9110	0.3945	0.9746	2.1632	0.8069	1.3021	1.0039	1.0224	1.0014	0.9841	1.0383	1.0702	1.0483	1.0448
1.0081	1.0075	1.0033	1.0021	1.0007	1.0269	1.0055	0.9870	1.0002	1.0166	1.0222	1.0130	1.0221	1.0240
1.0017	1.0025	1.0027	1.0017	1.0016	1.0541	0.9582	0.9329	0.9941	1.0058	1.0077	1.0063	1.0107	1.0114
1.0000	1.0005	1.0031	1.0023	1.0029	1.0384	0.9006	0.8792	0.9879	1.0034	1.0009	1.0049	1.0044	1.0046
1.0005	1.0000	1.0013	1.0015	1.0020	1.0118	0.8453	0.8846	0.9860	0.9997	0.9966	1.0022	1.0012	1.0013
1.0031	1.0013	1.0000	0.9997	1.0000	1.0010	0.9152	0.9406	0.9932	1.0013	1.0018	1.0019	1.0054	1.0060
1.0023	1.0015	0.9997	1.0000	1.0004	1.0191	0.7886	0.8952	0.9898	1.0007	1.0000	1.0035	1.0050	1.0056
1.0029	1.0020	1.0000	1.0004	1.0000	1.0155	0.7723	0.8444	0.9900	1.0019	1.0017	1.0046	1.0069	1.0075
1.0384	1.0118	1.0010	1.0191	1.0155	1.0000	0.8923	0.9403	1.0010	0.9988	0.9880	1.0024	0.9945	0.9952
0.9006	0.8453	0.9152	0.7886	0.7723	0.8923	1.0000	1.0028	0.9805	0.9496	0.9607	0.5792	0.8225	0.8272
0.8792	0.8846	0.9406	0.8952	0.8444	0.9403	1.0028	1.0000	0.9925	0.9703	0.9767	0.9185	0.9454	0.9467
0.9879	0.9860	0.9932	0.9898	0.9900	1.0010	0.9805	0.9925	1.0000	0.9972	0.9988	0.9908	0.9953	0.9960
1.0034	0.9997	1.0013	1.0007	1.0019	0.9988	0.9496	0.9703	0.9972	1.0000	0.9999	0.9983	1.0005	1.0010
1.0009	0.9966	1.0018	1.0000	1.0017	0.9880	0.9607	0.9767	0.9988	0.9999	1.0000	0.9954	0.9986	0.9990
1.0049	1.0022	1.0019	1.0035	1.0046	1.0024	0.5792	0.9185	0.9908	0.9983	0.9954	1.0000	0.9991	0.9990
1.0044	1.0012	1.0054	1.0050	1.0069	0.9945	0.8225	0.9454	0.9953	1.0005	0.9986	0.9991	1.0000	1.0000
1.0046	1.0013	1.0060	1.0056	1.0075	0.9952	0.8272	0.9467	0.9960	1.0010	0.9990	0.9990	1.0000	1.0000
CPU time = 0.0973 second													

able 33. SVD algorithm using 20 ground motions, 1,000,000 simulations, example 2, no acceleration

Demand Parameters													
Floor Acceleration $a_i(g)$						Drift Ratio $\delta_i(\%)$							
4	5	6	7	8	9	1	2	3	4	5	6	7	8
Ratio of means													
1.0003	1.0005	1.0003	1.0003	1.0001	0.9999	1.0000	1.0000	1.0000	1.0000	1.0001	1.0002	1.0004	1.0005
Ratio of correlation coefficients													
1.0100	1.0880	1.0107	0.7894	1.0410	0.9304	0.9999	0.9999	1.0032	1.0135	1.0072	0.9902	0.9917	0.9915
1.0002	1.0004	0.9998	1.0000	1.0000	0.9990	1.0022	1.0053	1.0004	0.9991	0.9989	0.9996	0.9991	0.9991
1.0001	1.0002	0.9998	1.0000	1.0001	0.9982	1.0031	1.0044	1.0002	1.0001	1.0002	1.0001	0.9999	0.9999
1.0000	1.0000	0.9997	0.9999	1.0001	0.9957	1.0058	1.0029	1.0001	1.0000	1.0000	0.9997	0.9995	0.9995
1.0000	1.0000	1.0000	1.0000	1.0001	0.9985	1.0114	1.0040	1.0005	1.0003	1.0004	0.9999	0.9998	0.9997
0.9997	1.0000	1.0000	1.0000	1.0000	1.0020	1.0091	1.0044	1.0004	1.0001	1.0003	1.0001	1.0000	1.0000
0.9999	1.0000	1.0000	1.0000	1.0000	0.9990	1.0196	1.0059	1.0004	1.0001	1.0003	0.9999	0.9998	0.9998
1.0001	1.0001	1.0000	1.0000	1.0000	0.9971	1.0267	1.0114	1.0004	1.0001	1.0003	0.9998	0.9997	0.9997
0.9957	0.9985	1.0020	0.9990	0.9971	1.0000	1.0176	1.0168	1.0060	1.0052	1.0081	1.0015	1.0030	1.0032
1.0058	1.0114	1.0091	1.0196	1.0267	1.0176	1.0000	1.0001	1.0023	1.0056	1.0034	1.0375	1.0132	1.0135
1.0029	1.0040	1.0044	1.0059	1.0114	1.0168	1.0001	1.0000	1.0008	1.0022	1.0012	1.0046	1.0023	1.0024
1.0001	1.0005	1.0004	1.0004	1.0004	1.0060	1.0023	1.0008	1.0000	1.0001	1.0000	1.0007	1.0003	1.0004
1.0000	1.0003	1.0001	1.0001	1.0001	1.0052	1.0056	1.0022	1.0001	1.0000	1.0000	1.0003	1.0002	1.0002
1.0000	1.0004	1.0003	1.0003	1.0003	1.0081	1.0034	1.0012	1.0000	1.0000	1.0000	1.0005	1.0003	1.0003
0.9997	0.9999	1.0001	0.9999	0.9998	1.0015	1.0375	1.0046	1.0007	1.0003	1.0005	1.0000	1.0000	1.0000
0.9995	0.9998	1.0000	0.9998	0.9997	1.0030	1.0132	1.0023	1.0003	1.0002	1.0003	1.0000	1.0000	1.0000
0.9995	0.9997	1.0000	0.9998	0.9997	1.0032	1.0135	1.0024	1.0004	1.0002	1.0003	1.0000	1.0000	1.0000
CPU time =3.3300 second													

Table 34. SVD algorithm using 20 ground motions, example 2, with acceleration

Demand Parameters													
Floor acceleration $a_i(g)$						Drift ratio $\delta_i(\%)$							
4	5	6	7	8	9	1	2	3	4	5	6	7	8
Ratio of means													
1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
Ratio of correlation coefficients													
1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
U time is 0.0686, 0.1096 and 4.9860 second for simulation size 100, 10000 and 1000000, respectively													

Appendix A

Algorithm based on Spectral Value Decomposition

Let $[Y]=[Y_1\ Y_2\ ..\ Y_n]$, a $m\times n$ matrix represents the m realizations of an n dimensional random variable $\widehat{Y}$ such that

$$\widehat{Y}=\begin{bmatrix}\widehat{Y_1}\\ \widehat{Y_2}\\ \cdot\\ \widehat{Y_n}\end{bmatrix}=[Y]^T \tag{A1}$$

Define $[G]=[G_1\ G_2\ ..\ G_n]$ such that

$$G_i=Y_i-\mu_{Y_i}\{1\} \tag{A2}$$

where μ_{Y_i} is the mean of the i^{th} column of $[Y]$ and $\{1\}$ is $m\times 1$ vector of ones. Matrix $\widehat{G}=[G]^T$ represents the random variable $\widehat{Y}$ with a zero mean. The statistical properties of $\widehat{G}$, for example, the covariance matrix $\Sigma\left(\widehat{G}\right)$ and the correlation matrix $R\left(\widehat{G}\right)$ are

$$\Sigma\left(\widehat{G}\right)=\frac{1}{m-1}[G]^T[G]=\frac{1}{m-1}\widehat{G}\widehat{G}^T \tag{A3}$$

$$R\left(\widehat{G}\right)=[\sigma_G]^{-1}\Sigma\left(\widehat{G}\right)[\sigma_G]^{-1} \tag{A4}$$

where $\left[\sigma_G^2\right]=[\sigma_G][\sigma_G]$ is a diagonal matrix of variances.

Let, $\widehat{\widehat{G}}=\left[\overline{G}\right]^T$ be the simulated random variable of M realizations with a specified mean and a specified covariance matrix (in other words, a specified variance and a specified correlation matrix). These specified statistics are the same as those of $\widehat{G}$. Here $\left[\overline{G}\right]$ is a $M\times n$ matrix and the specified mean of each vector is zero. This step is performed by generating the required set of independent random variables $\widehat{\widehat{U}}=\left[\overline{U}\right]^T$ of the same distribution with zero mean and unit variance followed by a linear transformation of the form

$$\widehat{\widehat{G}}=[T][J]\widehat{\widehat{U}} \tag{A5}$$

Here $[T]$ and $[J]$ are $n\times n$ matrices representing the linear transformation from the independent random variables to the correlated random variables. Using the property of linear transformation, the covariance matrix is given by

$$\Sigma\left(\widehat{\overline{G}}\right)=\frac{1}{M-1}\left[\overline{G}\right]^T\left[\overline{G}\right]=\frac{1}{M-1}\widehat{\overline{G}}\widehat{\overline{G}}^T=[TJ]\Sigma\left(\widehat{\overline{U}}\right)[TJ]^T \tag{A6}$$

Accordingly, the correlation matrix is

$$R\left(\widehat{\overline{G}}\right)=\left(\left[\sigma_{\overline{G}}\right]^{-1}[TJ]\right)\Sigma\left(\widehat{\overline{U}}\right)\left(\left[\sigma_{\overline{G}}\right]^{-1}[TJ]\right)^T \tag{A7}$$

and, $\left[\sigma_{\overline{G}}^2\right]=\left[\sigma_{\overline{G}}\right]\left[\sigma_{\overline{G}}\right]$ is a diagonal matrix of variances.

-o-

Near infinite number of simulations; no acceleration

For this case, $\Sigma\left(\widehat{\overline{U}}\right)$ is approximately an identity matrix of order n. Accordingly, Eq (A7) leads to

$$\begin{aligned}R\left(\widehat{\overline{G}}\right)&=[L][L]^T\\ [L]&=\left(\left[\sigma_{\overline{G}}\right]^{-1}[TJ]\right)\end{aligned} \tag{A8}$$

Using Eq (A3) and Eq (A4), the correlation matrix of the original data set may be expressed as

$$\begin{aligned}R\left(\widehat{G}\right)&=[K]^T[K]\\ [K]&=\left(\frac{1}{\sqrt{m-1}}[G][\sigma_G]^{-1}\right)\end{aligned} \tag{A9}$$

Since, $R\left(\widehat{G}\right)$ i) is a symmetric matrix and ii) can be expressed as $[K]^T[K]$, this is possible to show that all the eigenvalues of $R\left(\widehat{G}\right)$ will be non-negative. By spectral value decomposition

$$\begin{aligned}R\left(\widehat{G}\right)&=[A_{Gcor}][\lambda_{Gcor}][A_{Gcor}]^T=[L][L]^T\\ [L]&=[A_{Gcor}][\lambda_{Gcor}]^{1/2}\end{aligned} \tag{A10}$$

Since, the simulated random variables have the same correlation matrix as that of the original data set, and by equating $[L]$ from Eq (A8) and Eq (A10), it can be shown that

$$[TJ]=\left[\sigma_{\overline{G}}\right][A_{Gcor}][\lambda_{Gcor}]^{1/2} \tag{A11}$$

Substituting, Eq (A11) into Eq (A5), taking the transpose, and noting $\widehat{\overline{G}}=\left[\overline{G}\right]^T$ and $[\sigma_G]=\left[\sigma_{\overline{G}}\right]$, it may be shown that

$$\left[\overline{G}\right]=\left[\overline{U}\right]\left[\lambda_{Gcor}\right]^{1/2}\left[A_{Gcor}\right]^{T}\left[\sigma_{G}\right] \tag{A12}$$

This form is identical to the PACT algorithm if the Cholesky decomposition $R\left(\widehat{G}\right)=[L][L]^{T}$ exists, where $[L]^{T}$ in the Cholesky decomposition is equivalent to $\left[\lambda_{Gcor}\right]^{1/2}\left[A_{Gcor}\right]^{T}$ in the spectral value decomposition.

Small number of simulations; with acceleration

In this case $\Sigma\left(\widehat{\widehat{U}}\right)$ will be considerably different from the identity matrix of order n. First, we subtract the sample mean from $\widehat{\widehat{U}}$ and compute $\Sigma\left(\widehat{\widehat{U}}\right)$. Assuming the number of simulation to be greater than the number of demand parameters, that is, $M>n$, spectral value decomposition shows that

$$\Sigma\left(\widehat{\widehat{U}}\right)=\left[A_{\mathrm{cov}U}\right]\left[\lambda_{\mathrm{cov}U}\right]\left[A_{\mathrm{cov}U}\right]^{T} \tag{A13}$$

Factorization of the form of Eq (A13) is possible because $M>n$ guarantees that all the eigenvalues of $\Sigma\left(\widehat{\widehat{U}}\right)$ are non-negative. Accordingly, the correlation matrix of the simulated random variables is given by [from Eq(A7)]

$$R\left(\widehat{\widehat{G}}\right)=\left(\left[\sigma_{\overline{G}}\right]^{-1}[T]\right)\left([J]\Sigma\left(\widehat{\widehat{U}}\right)[J]^{T}\right)\left(\left[\sigma_{\overline{G}}\right]^{-1}[T]\right)^{T} \tag{A14}$$

Next we select $[J]$ such that $\left([J]\Sigma\left(\widehat{\widehat{U}}\right)[J]^{T}\right)$ becomes an identity matrix of order n. This is possible when

$$[J]=\left[\lambda_{\mathrm{cov}U}\right]^{-1/2}\left[A_{\mathrm{cov}U}\right]^{T} \tag{A15}$$

Accordingly, Eq (A14) is reduced to

$$\begin{aligned} R\left(\widehat{\widehat{G}}\right)&=[L][L]^{T}\\ [L]&=\left(\left[\sigma_{\overline{G}}\right]^{-1}[T]\right) \end{aligned} \tag{A16}$$

Since, the simulated random variables have the same correlation matrix as that of the original data set, and equating $[L]$ from Eq (A16) and Eq (A10), it may be shown that

$$[T]=\left[\sigma_{\overline{G}}\right]\left[A_{Gcor}\right]\left[\lambda_{Gcor}\right]^{1/2} \tag{A17}$$

Substituting, Eq (A17) and Eq (A15) into Eq (A5), taking transpose and noting $\widehat{\widehat{G}}=\left[\overline{G}\right]^{T}$ and $\left[\sigma_{G}\right]=\left[\sigma_{\overline{G}}\right]$, it may be shown that

$$\left[\overline{G}\right]=\left[\overline{U}\right]\left[A_{\mathrm{cov}U}\right]\left[\lambda_{\mathrm{cov}U}\right]^{-1/2}\left[\lambda_{Gcor}\right]^{1/2}\left[A_{Gcor}\right]^{T}\left[\sigma_{G}\right] \tag{A18}$$

This procedure preserves the mean, variance and correlation matrix regardless of the simulation size if $M > n$, which is not a constraint.

-o-

Once, $\left[\overline{G}\right]$ is computed, $\left[\overline{Y}\right]$ may be computed by adding back the mean as

$$\overline{Y}_i = \overline{G}_i + \mu_{Y_i}\left\{1\right\} \tag{A19}$$

where $\left\{1\right\}$ is a $M \times 1$ vector of ones.

This completes the simulation of the multivariate normal variable $\widehat{Y}$ with specified mean, variance and correlation matrix. The user can choose to accelerate the recovery of the underlying statistics. Irrespective of the simulation size (or number of simulations), the SVD algorithm returns results similar to the *cholcov* routine in PACT if the acceleration option is not selected. For a small number of simulations, the use of the acceleration option appears initially to violate some basic rules of statistics because instead of performing a linear transformation on simulated standard normal variables $\widehat{\widehat{U}}$, a linear transformation is performed on other simulated random variables $\widehat{\widehat{V}}$. Here $\widehat{\widehat{V}}$ is linearly related to $\widehat{\widehat{U}}$ (after subtracting the sample mean) such that $\widehat{\widehat{V}} = \left[J\right]\widehat{\widehat{U}}$ where $\left[J\right]$ is computed from the spectral value decomposition of $\Sigma\left(\widehat{\widehat{U}}\right)$ (after subtracting the sample mean) per Eq (A15). The transformation $\left[J\right]$ is such that $\Sigma\left(\widehat{\widehat{V}}\right) = \Sigma\left(\widehat{U_\infty}\right)$ and where $\widehat{U_\infty}$ is the simulated standard normal variables for an infinite number of realizations (or simulations). The degree to which the acceleration option violates fundamental rules of statistics can be measured by testing of the distribution of $\widehat{\widehat{V}}$ with respect to the standard normal distribution. Such testing does not require the original data set to be known (assuming $M > n$).